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2006 J. Phys. A: Math. Gen. 39 13455

(http://iopscience.iop.org/0305-4470/39/43/005)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 13455–13463 doi:10.1088/0305-4470/39/43/005

Systematical approach to the exact solution of the Dirac equation for a deformed form of the Woods–Saxon potential

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Received 23 February 2006, in final form 6 September 2006
Published 11 October 2006
Online at stacks.iop.org/JPhysA/39/13455

Abstract

The exact solution of the Dirac equation for a deformed form of the Woods–Saxon potential is obtained for the s-wave relativistic energy spectrum. The energy eigenvalues and two-component spinor wavefunctions are derived analytically by using a systematical method which is called Nikiforov–Uvarov. It is seen that the energy eigenvalues and the wavefunctions strongly depend on the parameters of the potential. In addition, it is also shown that the non-relativistic limit can be reached easily and directly for a special case of the standard Woods–Saxon potential.

PACS numbers: 03.65.Ge, 03.65.Pm, 02.30.Gp, 31.30.Jv

1. Introduction

The Woods–Saxon potential and its various modifications have played in recent years an important role in understanding the energy level spacing, particle number dependence of energy quantities and universal properties of the electron distributions in atoms, nuclei and atomic clusters [1–3]. It has been used in the central part of the interaction of neutron with one heavy-ion nucleus and also for the optical potential model [4]. The exact solutions of the Schrödinger-like equations used in optical model calculations have not known in analytical form until four decades ago. The first analytical solution of the Schrödinger equation for the Woods–Saxon potential was published by Bose [5], who formulated the problem of the construction of solvable one-variable Schrödinger potentials, while the more general case was generalized by Bencze [6], for a sum of the Woods–Saxon potential and its derivative form. After this generalization, the bound-state energies were obtained by solving the transcendental equation which can be obtained from the explicit expression for the S-matrix element. According to this solution, the differential cross section in $^{16}O+^{12}C$ elastic scattering

has been analysed in some energies by using this potential [7]. The relativistic Dirac-oscillator and Dirac-exponential-type potential problems have already been established by adding an off-diagonal linear radial term to the Dirac operator a long time ago [8, 9]. Recently, the relativistic bound-state energies and its eigenfunctions for the triaxial and axially deformed harmonic oscillators have been derived as well [10]. In addition, a mixture of the Dirac oscillator (tensor potential) with vector and scalar harmonic oscillator potentials has been solved analytically for the general case [11].

Furthermore, only a few articles for the relativistic problems have been written on the Dirac equation with the exponential-type potential. The Dirac equation has been solved by making use of two-component spinors for the exponential-type potentials such as Woods–Saxon and Hulthén potentials for a special case. Kennedy has studied the generalized approach to the Woods–Saxon potential and obtained the scattering and bound-state solutions of the one-dimensional Dirac equation. However, more realistic cases have not been discussed in this study [12]. Alhaidari has introduced a new formalism to the definition of the radial Dirac equation and solved for a class of shape-invariant potentials [13–15]. The main point in the formalism is that two coupled first-order differential equations coming from the radial Dirac equation generate Schrödinger-like equations for two spinor components. The solution method used in these studies has been based on hypergeometric functions and the obtained wavefunctions have been given in terms of these functions. This selection is very useful to understand the wavefunction solution but the energy spectrum is very confused for obtaining any quantum states.

Following the procedure given in [13], we present a new systematical approach to solve the Dirac–Woods–Saxon problem by means of the Nikiforov–Uvarov (NU) method [16]. The wavefunction solutions are given in terms of Jacobi polynomials and then the energy eigenvalues are obtained easily. From this point of view, we can say that the present work provides a short-cut solution procedure on this problem and is useful to obtain the energy spectrum of a Dirac particle. In addition, the non-relativistic limit is discussed for a special case of the standard Woods–Saxon potential.

The paper is structured as follows: in section 2, we briefly introduce an overview of the technical details of the formalism improved by Alhaidari. After that, the basic concepts of the Nikiforov–Uvarov method are given in the same section to solve the Dirac–Woods–Saxon problem. Section 3 is devoted to the solution of the problem to obtain the energy eigenvalues and eigenfunctions by applying the NU method. The paper is concluded with a short summary in section 4

2. Formalism and method

We first introduced Alhaidari's formalism to solve the Dirac equation which interacts with a spherically symmetric potential. Later, the fundamental mathematical relations of the NU method are summarized to obtain the solution of the Schrödinger-like equations easily and systematically.

2.1. Overview of the formalism

The basic idea of Alhaidari's formalism is to write the relativistic Hamiltonian for a Dirac spinor coupled to a four-component potential $(A_0, \vec{A})$. After using the spherically symmetric characteristic of the potential, free Dirac equation transforms to the matrix representation of the Dirac Hamiltonian (see [13] for more detail). For convenience, if atomic units are selected

as $m = e = \hbar = 1$ and the speed of light c is matched with α^{-1}, the Hamiltonian for a Dirac spinor in four-component potential $(A_0, \vec{A})$ coupled non-minimally can be written as follows:

$$H = \begin{pmatrix} 1 + \alpha A_0 & -\mathrm{i}\alpha\vec{\sigma}\cdot\vec{\nabla} + \mathrm{i}\alpha\vec{\sigma}\cdot\vec{A} \\ -\mathrm{i}\alpha\vec{\sigma}\cdot\vec{\nabla} - \mathrm{i}\alpha\vec{\sigma}\cdot\vec{A} & -1 + \alpha A_0 \end{pmatrix} \tag{1}$$

where α is the Compton wavelength and $\vec{\sigma}$ is three 2×2 Pauli spin matrices. Taking into account the spherically symmetric cases and writing $(A_0, \vec{A})$ as $(\alpha V(r), \hat{r}W(r))$, the two-component Dirac equation is obtained as

$$H = \begin{pmatrix} 1 + \alpha^2 V(r) - E_R & \alpha\left[\frac{\kappa}{r} + W(r) - \frac{\mathrm{d}}{\mathrm{d}r}\right] \\ \alpha\left[\frac{\kappa}{r} + W(r) + \frac{\mathrm{d}}{\mathrm{d}r}\right] & -1 + \alpha^2 V(r) - E_R \end{pmatrix} \begin{pmatrix} g(r) \\ f(r) \end{pmatrix} = 0 \tag{2}$$

where $f(r)$ and $g(r)$ are the integrable functions, E_R is the relativistic energy and κ is the spin–orbit coupling parameter defined as $\kappa = \pm(j + 1/2) = \pm 1, \pm 2, \ldots$, for $l = j \pm 1/2$. However, the current problem is analytically solvable only for $\ell = 0$ (s-states). In addition, $V(r)$ and $W(r)$ are the even and odd components of the relativistic potential, respectively. For a given value of the spin–orbit coupling parameter κ, the Schrödinger-like requirement relates to the following potential function: $W(r) = \frac{1}{\xi}V(r) - \frac{\kappa}{r}$, where ξ is a real parameter and $V(r)$ does not depend on the κ parameter. In order to obtain the Schrödinger-like equation from the formalism proposed by Alhaidari, a global unitary transformation which eliminates the first derivative is used. Thus, $\mathcal{U}(\eta) = \exp\left(\frac{\mathrm{i}}{2}\alpha\eta\sigma_2\right)$ is applied in equation (2). η is a real constant and σ_2 is the 2×2 Pauli matrix which defines the two radial spinor components in terms of the other,

$$\phi^{\mp}(r) = \frac{\alpha}{C \pm E_R}\left[-\xi \pm \frac{C}{\xi}V(r) + \frac{\mathrm{d}}{\mathrm{d}r}\right]\phi^{\pm}(r), \tag{3}$$

with $C = \cos(\alpha\eta) = \sqrt{1 - (\alpha\xi)^2} > 0$,

$$\begin{pmatrix} \phi^{+}(r) \\ \phi^{-}(r) \end{pmatrix} = \mathcal{U}\begin{pmatrix} g(r) \\ f(r) \end{pmatrix}. \tag{4}$$

Here, $\phi^{\pm}(r)$ is the upper or lower spinor components, respectively. It is emphasized that equation (3) with the top and bottom signs is not valid for negative and positive energy solutions, respectively. The top and bottom signs in front of E_R in equation (3) are not allowed to take the values $-C$ and $+C$. Because these values are elements of the negative and positive energy spectra, respectively. Substituting these values into the radial Dirac equation (equation (2)), we obtain the Schrödinger-like second-order differential equation in terms of the lower and upper spinor components as follows:

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{C^2}{\xi^2}V^2 + 2E_R V \mp \frac{C}{\xi}\frac{\mathrm{d}V}{\mathrm{d}r} - \frac{E_R^2 - 1}{\alpha^2}\right]\phi^{\pm}(r) = 0, \tag{5}$$

where the '+' sign belongs to the upper spinor component, while the other sign corresponds to the lower one.

2.2. Basic concepts of the method

The solution of the Schrödinger-like second-order differential equations plays an essential role in studying many important problems of theoretical physics. In this regard, the NU method can be used to solve these types of equations with an appropriate coordinate transformation $s = s(r)$ [16]:

$$\psi''(s) + \frac{\widetilde{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\widetilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0 \tag{6}$$

where $\sigma(s)$ and $\widetilde{\sigma}(s)$ are the polynomials with at most second degree, and $\widetilde{\tau}(s)$ is a first-degree polynomial. It is of fundamental importance in the study of particular special orthogonal polynomials [17]. These polynomials try to reduce equation (6) to a simple form by taking $\psi(s) = \phi(s)y(s)$ and choosing an appropriate function $\phi(s)$. Consequently, equation (6) can be reduced to an equation of hypergeometric type:

$$\sigma(s)y''(s) + \tau(s)y'(s) + \lambda y(s) = 0, \tag{7}$$

where $\tau(s) = \widetilde{\tau}(s) + 2\pi(s)$ (its derivative must be negative) and λ is a constant given in the form

$$\lambda = \lambda_n = -n\tau' - \frac{n(n-1)}{2}\sigma'' \qquad (n = 0, 1, 2, \ldots). \tag{8}$$

Here, λ or λ_n are obtained from a particular solution of the form $y(s) = y_n(s)$, which is a polynomial of degree n. $y_n(s)$ is the hypergeometric-type function whose polynomial solutions are given by the Rodrigues relation

$$y_n(s) = \frac{B_n}{\rho(s)}\frac{\mathrm{d}^n}{\mathrm{d}s^n}[\sigma^n(s)\rho(s)], \tag{9}$$

where B_n is the normalization constant and the weight function $\rho(s)$ must satisfy the condition

$$[\sigma(s)\rho(s)]' = \tau(s)\rho(s). \tag{10}$$

To determine the weight function given in equation (10), we must immediately obtain the following polynomial $\pi(s)$:

$$\pi = \frac{\sigma' - \widetilde{\tau}}{2} \pm \sqrt{\left(\frac{\sigma' - \widetilde{\tau}}{2}\right)^2 - \widetilde{\sigma} + k\sigma}. \tag{11}$$

In principle, the expression under the square root sign in equation (11) can be arranged as the square of a polynomial. This is possible only if its discriminant is zero. In this case, an equation for k is obtained. After solving this equation, the obtained values of k are included in the NU method and here there is a relationship with λ of k so that $k = \lambda - \pi'(s)$. After this point, an appropriate $\phi(s)$ can be invented from $\phi(s)'/\phi(s) = \pi(s)/\sigma(s)$.

3. Deformed form of the Woods–Saxon potential

The interaction among nuclei is commonly described by using a potential which consists of the Coulomb and the nuclear potentials. It is usually taken in the form of the Woods–Saxon potential. Here, we take into account the following deformed form for the Woods–Saxon potential which is specified by the parameter 'q',

$$V(r) = -\frac{qV_0}{q + \mathrm{e}^{(\frac{r-R_0}{b})}}, \tag{12}$$

where V_0 is the potential depth, R_0 is the width of the potential, b is thickness of the surface which is usually adjusted to the experimental values of ionization energies and q is a real positive parameter which is responsible for the deformation of the Woods–Saxon potential. It should also be noted that this problem is, in fact, equivalent to the standard Woods–Saxon problem under the radial coordinate transformation $r \to r + a$, where the displacement parameter a satisfies the expression $\exp(a/b) = q$. Under this transformation, $V(r)$ in

equation (12) turns into the standard Woods–Saxon potential [18, 19]. After substituting the potential into equation (5), we obtain an equation for the upper spinor component:

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{C^2}{\xi^2}\left(\frac{qV_0}{q+\mathrm{e}^{(\frac{r-R_0}{b})}}\right)^2-\frac{2qE_RV_0}{q+\mathrm{e}^{(\frac{r-R_0}{b})}}-\frac{qCV_0}{\xi b}\frac{\mathrm{e}^{(\frac{r-R_0}{b})}}{\left(q+\mathrm{e}^{(\frac{r-R_0}{b})}\right)^2}-\frac{E_R^2-1}{\alpha^2}\right]\phi^+(r)=0,$$

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{qCV_0}{\xi b}\frac{\frac{qCV_0b}{\xi}-\mathrm{e}^{(\frac{r-R_0}{b})}}{\left(q+\mathrm{e}^{(\frac{r-R_0}{b})}\right)^2}-\frac{2qE_RV_0}{q+\mathrm{e}^{(\frac{r-R_0}{b})}}-\frac{E_R^2-1}{\alpha^2}\right]\phi^+(r)=0. \tag{13}$$

In order to apply the NU method, we rewrite equation (13) by using a new variable of the form $s=\left(q+\mathrm{e}^{\frac{r-R_0}{b}}\right)^{-1}$,

$$\left[-\frac{s(1-qs)}{b}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{s(1-qs)}{b}\frac{\mathrm{d}}{\mathrm{d}s}\right)+\frac{q^2C^2V_0^2}{\xi^2}s^2\right.$$

$$\left.-2qE_RV_0s-\frac{qCV_0}{\xi b}s(1-qs)-\frac{E_R^2-1}{\alpha^2}\right]\phi^+(s)=0. \tag{14}$$

By introducing the following dimensionless parameters

$$\varepsilon=\left(\frac{E_R^2-1}{\alpha^2}\right)b^2,\qquad \beta=2qE_RV_0b^2,\qquad \gamma=\frac{qCV_0b}{\xi}, \tag{15}$$

we reach the following hypergeometric-type equation defined in equation (6):

$$\frac{\mathrm{d}^2\phi^+(s)}{\mathrm{d}s^2}+\frac{1-2qs}{s(1-qs)}\frac{\mathrm{d}\phi^+(s)}{\mathrm{d}s}+\frac{1}{s^2(1-qs)^2}\times[-s^2(\gamma^2+\gamma q)+s(\beta+\gamma)+\varepsilon]\phi^+(s)=0. \tag{16}$$

After comparing equation (16) with equation (6), we obtain the corresponding polynomials:

$$\widetilde{\tau}(s)=1-2qs,\qquad \sigma(s)=s(1-qs),\qquad \widetilde{\sigma}(s)=-s^2(\gamma^2+\gamma q)+s(\beta+\gamma)+\varepsilon. \tag{17}$$

Substituting these polynomials into equation (11), we organize the polynomial $\pi(s)$ as follows:

$$\pi(s)=\pm\sqrt{(\gamma^2+\gamma q-kq)s^2+(k-\beta-\gamma)s-\varepsilon}, \tag{18}$$

with $\sigma'(s)=1-2qs$. It is taken into consideration that the discriminant of the second-order equation under the square root sign has to be zero. Hence, the expected roots are obtained as $k_\pm=\beta+\gamma+2\varepsilon q\pm2\sqrt{\varepsilon^2q^2+\beta\varepsilon q-\varepsilon\gamma^2}$. In this case, substituting these values for each k into equation (18), the possible solutions are obtained for $\pi(s)$:

$$\pi(s)=\pm\mathrm{i}\begin{cases}\left(q\sqrt{\varepsilon}+\sqrt{\varepsilon q^2+\beta q-\gamma^2}\right)s-\sqrt{\varepsilon},\\ \text{for}\quad k_+=\beta+\gamma+2\varepsilon q+2\sqrt{\varepsilon^2q^2+\beta\varepsilon q-\varepsilon\gamma^2},\\ \left(q\sqrt{\varepsilon}-\sqrt{\varepsilon q^2+\beta q-\gamma^2}\right)s-\sqrt{\varepsilon},\\ \text{for}\quad k_-=\beta+\gamma+2\varepsilon q-2\sqrt{\varepsilon^2q^2+\beta\varepsilon q-\varepsilon\gamma^2}.\end{cases} \tag{19}$$

From the four possible forms of the polynomial $\pi(s)$, we have to select an appropriate one. In this case, the derivative of $\tau(s)$ takes a negative value. Therefore, the function $\tau(s)$ satisfies

the following equalities:

$$\begin{aligned}\tau(s) &= 1 + 2\mathrm{i}\sqrt{\varepsilon} - s\big(2q + 2\mathrm{i}\big[q\sqrt{\varepsilon} + \sqrt{\varepsilon q^2 + \beta q - \gamma^2}\big]\big),\\ \tau'(s) &= -\big(2q + 2\mathrm{i}\big[q\sqrt{\varepsilon} + \sqrt{\varepsilon q^2 + \beta q - \gamma^2}\big]\big) < 0.\end{aligned} \tag{20}$$

In the present case, the polynomial $\pi(s)$ becomes

$$\pi(s) = -\mathrm{i}\big[\big(q\sqrt{\varepsilon} + \sqrt{\varepsilon q^2 + \beta q - \gamma^2}\big)s - \sqrt{\varepsilon}\,\big]. \tag{21}$$

From $k = \lambda - \pi'(s)$ and also equation (8), we obtain respectively

$$\lambda = \beta + \gamma + 2\varepsilon q + 2\sqrt{\varepsilon^2 q^2 + \beta\varepsilon q - \varepsilon\gamma^2} - \mathrm{i}\big(q\sqrt{\varepsilon} + \sqrt{\varepsilon q^2 + \beta q - \gamma^2}\big), \tag{22}$$

$$\lambda_n = n^2 q + nq + 2n\mathrm{i}\big(q\sqrt{\varepsilon} + \sqrt{\varepsilon q^2 + \beta q - \gamma^2}\big). \tag{23}$$

After comparing equation (22) and equation (23), the obtained result leads to

$$\begin{aligned}&\big(q\sqrt{\varepsilon} + \sqrt{\varepsilon q^2 + \beta q - \gamma^2}\big)^2(1 + 2n + 4\varepsilon) - 4\sqrt{\varepsilon}(nq + n^2 q - \beta - \gamma)\\ &\qquad \times \big(q\sqrt{\varepsilon} + \sqrt{\varepsilon q^2 + \beta q - \gamma^2}\big) + (nq + n^2 q - \beta - \gamma)^2 = 0,\end{aligned} \tag{24}$$

which is the eigenvalue condition for the κ-dependent relativistic energy eigenvalue E_R of the Dirac particle. Explicit solutions of equation (24), giving E_R in terms of the parameters of the potential, can be determined numerically. It is especially noted that the relativistic energy spectrum is normally κ-dependent but it is not here simply because of considering only the s-wave problem ($\ell = 0$ or, equivalently, $\kappa = -1$) which is numerically solvable. To have a physical result, the expressions under the square roots must be positive. n is a positive integer defined in the interval of $n_{\max} \geqslant n \geqslant 0$ and is called the radial quantum number.

When we deal with equation (13), it can easily seen that, in the non-relativistic limit $\alpha \to 0$, the relativistic energy is a limit of the non-relativistic energy, $E_R \approx 1 + \alpha^2 E_{\mathrm{NR}}$, where E_{NR} is the non-relativistic energy. The wave equation is reduced to the following form, choosing $q = 1$:

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\gamma}{b^2}\frac{\gamma - \mathrm{e}^{(\frac{r-R_0}{b})}}{\big(1 + \mathrm{e}^{(\frac{r-R_0}{b})}\big)^2} - \frac{2V_0}{1 + \mathrm{e}^{(\frac{r-R_0}{b})}} - 2E_{\mathrm{NR}}\right]\phi^+(r) = 0. \tag{25}$$

To obtain a more suitable case, we can use the following form after taking $\gamma = -1$:

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} - 2\frac{V_0 - 1/2b^2}{1 + \mathrm{e}^{(\frac{r-R_0}{b})}} - 2E_{\mathrm{NR}}\right]\phi^+(r) = 0, \tag{26}$$

which is in the form of the Schrödinger equation for the new type s-wave non-relativistic Woods–Saxon potential. The corresponding energy spectrum for a bound state has been already given in [18] by the transcendental equation:

$$\begin{aligned}&\sqrt{2(V_0 + E_{\mathrm{NR}})}R_0 - \sum_{n=0}^{\infty}\left(\tan^{-1}\frac{2\sqrt{2b^2(V_0 + E_{\mathrm{NR}})}}{n+1} - 2\tan^{-1}\frac{\sqrt{2b^2(V_0 + E_{\mathrm{NR}})}}{n + 1 + \mathrm{i}\sqrt{2b^2 E_{\mathrm{NR}}}}\right)\\ &\qquad + \tan^{-1}\frac{\sqrt{2b^2(V_0 + E_{\mathrm{NR}})}}{\mathrm{i}\sqrt{2b^2 E_{\mathrm{NR}}}} = \pm(n+1)\pi,\end{aligned} \tag{27}$$

where the transformation $V_0 \to V_0 - 1/2b^2$ is applied for the convenience and the index n relates to the radial quantum number ($n = 0, 1, 2, 3, \ldots$). In order to obtain the relativistic energy spectrum directly, considering the relativistic equation (13) and the non-relativistic equation (27) for the case of $\gamma = -1$, we can propose the relevant parameter map:

$$b \to b, \qquad R_0 \to R_0, \qquad V_0 - 1/2b^2 \to E_R V_0 - 1/2b^2, \qquad E_{\mathrm{NR}} \to \big(E_R^2 - 1\big)/2\alpha^2. \tag{28}$$

Using the map between the parameters of the two equations, the resulting upper relativistic energy spectrum is found as follows:

$$\frac{R_0}{\alpha b}\sqrt{\left(E_R^2-1\right)b^2+2E_RV_0\alpha^2b^2-\alpha^2}-\sum_{n=0}^{\infty}\left(\tan^{-1}\frac{2\sqrt{\left(E_R^2-1\right)b^2+2E_RV_0\alpha^2b^2-\alpha^2}}{\alpha(n+1)}\right.$$
$$\left.-2\tan^{-1}\frac{\sqrt{\left(E_R^2-1\right)b^2+2E_RV_0\alpha^2b^2-\alpha^2}}{\alpha(n+1)+\mathrm{i}b\sqrt{E_R^2-1}}\right)$$
$$+\tan^{-1}\frac{\sqrt{\left(E_R^2-1\right)b^2+2E_RV_0\alpha^2b^2-\alpha^2}}{\mathrm{i}b\sqrt{E_R^2-1}}=\pm(n+1)\pi. \tag{29}$$

Equation (29) indicates that one deals with a family of the Woods–Saxon potential for the s-states and can also be used to describe the single-particle motion in nuclei.

Let us now find the corresponding wavefunctions. According to the NU method, the polynomial solutions of the hypergeometric function $y(s)$ depend on the determination of weight function $\rho(s)$ satisfying the differential equation $[\sigma(s)\rho(s)]'=\tau(s)\rho(s)$. Thus, $\rho(s)$ is calculated as

$$\rho(s)=(1-qs)^{2\mathrm{i}\nu}\,s^{2\mathrm{i}\sqrt{\varepsilon}}, \tag{30}$$

where $\nu=\sqrt{\varepsilon+\beta/q-(\gamma/q)^2}$. Substituting into the Rodrigues relation given in equation (9), the wavefunctions are obtained in the following form:

$$y_{nq}(s)=A_n\,(1-qs)^{-2\mathrm{i}\nu}\,s^{-2\mathrm{i}\sqrt{\varepsilon}}\frac{\mathrm{d}^n}{\mathrm{d}s^n}[(1-qs)^{n+2\mathrm{i}\nu}s^{n+2\mathrm{i}\sqrt{\varepsilon}}], \tag{31}$$

where A_n is the normalization constant. Taking $q=1$, the polynomial solutions of $y_n(s)$ are expressed in terms of the Jacobi polynomials, which is one of the orthogonal polynomials. In this case, the weight function is $(1-s)^{2\mathrm{i}\nu}s^{2\mathrm{i}\sqrt{\varepsilon}}$ and equation (31) is reduced to $\sim P_n^{(2\mathrm{i}\sqrt{\varepsilon},2\mathrm{i}\nu)}(1-2s)$ [17]. After substituting $\pi(s)$ and $\sigma(s)$ into the expression $\phi(s)'/\phi(s)=\pi(s)/\sigma(s)$, the other part of the wavefunction is found as

$$\phi(s)=(1-qs)^{\mathrm{i}\nu}s^{\mathrm{i}\sqrt{\varepsilon}}. \tag{32}$$

We write the upper spinor component in terms of the Jacobi polynomials

$$\phi_n^+(s)=B_ns^{\mathrm{i}\sqrt{\varepsilon}}(1-s)^{\mathrm{i}\nu}P_n^{(2\mathrm{i}\sqrt{\varepsilon},2\mathrm{i}\nu)}(1-2s), \tag{33}$$

where B_n is a normalization constant. This equation satisfies the following boundary conditions:

$$\phi_n^+(s)=0 \qquad \text{at} \quad s=0 \qquad (r\to\infty), \tag{34}$$
$$\phi_n^+(s)=0 \qquad \text{at} \quad s=1 \qquad (r=0). \tag{35}$$

The last condition is valid for realistic nuclei because the radius R_0 is very larger than the diffusivity b, i.e., $R_0\gg b$. The lower component of the spinor wavefunction can also be obtained by substituting equation (33) into equation (3). We should then solve the following equation:

$$\phi_n^-(s)=\frac{\alpha}{C+E_R}\left[-\xi+\frac{C}{\xi}V(s)-\frac{s(1-s)}{b}\frac{\mathrm{d}}{\mathrm{d}s}\right]\phi_n^+(s). \tag{36}$$

where $E_R \neq C$. This is possible if a new variable is introduced as $x = 1 - 2s$. Now, the equation of the lower spinor component has been transformed into the following form:

$$\phi_n^-(s) = \frac{\alpha}{C + E_{Rn}^{\pm}} \left[-\xi - \frac{V_0 C(1-x)}{2\xi} + \frac{(1-x)(1+x)}{2b} \frac{\mathrm{d}}{\mathrm{d}x} \right] \phi_n^+(s), \tag{37}$$

with

$$\phi_n^+(s) = C_n (1-x)^{\mathrm{i}\sqrt{\varepsilon}} (1+x)^{\mathrm{i}\nu} P_n^{(2\mathrm{i}\sqrt{\varepsilon}, 2\mathrm{i}\nu)}(x), \tag{38}$$

where C_n is the normalization constant and its value is equal to $B_n 2^{-\mathrm{i}\nu - \mathrm{i}\sqrt{\varepsilon}}$. If the following recursion relations and the differential formula satisfied by the Jacobi polynomials [20] are included to the solution

$$\begin{aligned}
(1+x) P_n^{(\mu,\varrho)}(x) &= \frac{2}{2n+\mu+\varrho+1} \left[(n+\varrho) P_n^{(\mu,\varrho-1)}(x) + (n+1) P_{n+1}^{(\mu,\varrho-1)}(x) \right], \\
(1-x) P_n^{(\mu,\varrho)}(x) &= \frac{2}{2n+\mu+\varrho+1} \left[(n+\mu) P_n^{(\mu-1,\varrho)}(x) - (n+1) P_{n+1}^{(\mu-1,\varrho)}(x) \right], \\
(1-x^2) \frac{\mathrm{d}P_n^{(\mu,\varrho)}}{\mathrm{d}x}(x) &= -n \left(x + \frac{\varrho - \mu}{2n+\mu+\varrho} \right) P_n^{(\mu,\varrho)}(x) + 2 \frac{(n+\mu)(n+\varrho)}{2n+\mu+\varrho} P_{n-1}^{(\mu,\varrho)}(x), \\
P_n^{(\mu,\varrho)}(x) &= \frac{n+\mu+\varrho+1}{2n+\mu+\varrho+1} P_n^{(\mu,\varrho+1)}(x) + \frac{n+\mu}{2n+\mu+\varrho+1} P_{n-1}^{(\mu,\varrho+1)}(x),
\end{aligned} \tag{39}$$

we obtained the lower spinor component in terms of the Jacobi polynomials as a function of s

$$\begin{aligned}
\phi_n^-(s) = \frac{\alpha}{C + E_R} (1-s)^{\mathrm{i}\nu} s^{\mathrm{i}\sqrt{\varepsilon}} \Bigg\{ & P_n^{(2\mathrm{i}\sqrt{\varepsilon}, 2\mathrm{i}\nu)}(1-2s) - \frac{\mathrm{i}\sqrt{\varepsilon}}{b} (1-s) P_n^{(2\mathrm{i}\sqrt{\varepsilon}, 2\mathrm{i}\nu)}(1-2s) \\
& + 2Ls P_n^{(2\mathrm{i}\sqrt{\varepsilon}, 2\mathrm{i}\nu)}(1-2s) - \frac{1}{b}(1-s)s \frac{\mathrm{d}P_n^{(2\mathrm{i}\sqrt{\varepsilon}, 2\mathrm{i}\nu)}(1-2s)}{\mathrm{d}s} \Bigg\},
\end{aligned} \tag{40}$$

where $L = \mathrm{i}\nu/2b - CV_0/\xi$.

4. Conclusion

We have solved the Dirac equation for the deformed form of the Woods–Saxon potential following a formalism introduced by Alhaidari. The NU method is used to obtain a systematical solution in the Dirac–Woods–Saxon problem. The energy spectrum of the bound states is analytically obtained and two-component spinor eigenfunctions are written in terms of the Jacobi polynomials. It is seen that the energy eigenvalues are a function of the parameter q and the solution space splits into two distinct subspaces. We have seen that the non-relativistic limit of the Dirac equation can be obtained easily.

Acknowledgments

The authors are indebted to A D Alhaidari for useful comments on the original version of the manuscript. This research was partially supported by the Scientific and Technological Research Council of Turkey.

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